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Relations and Functions

TOPIC 1 Relations, Domain, Codomain and Range of a Relation, Functions, Domain, Codomain and Range of a Function

1. Let R_1 and R_2 be two relations defined as follows :

$R_1 = \{(a, b) \in \mathbf{R}^2 : a^2 + b^2 \in Q\}$ and

$R_2 = \{(a, b) \in \mathbf{R}^2 : a^2 + b^2 \notin Q\}$, where Q is the set of all rational numbers. Then : **[Sep. 03, 2020 (II)]**

(a) Neither R_1 nor R_2 is transitive.
(b) R_2 is transitive but R_1 is not transitive.
(c) R_1 is transitive but R_2 is not transitive.
(d) R_1 and R_2 are both transitive.

2. The domain of the function $f(x) = \sin^{-1}\left(\frac{|x|+5}{x^2+1}\right)$ is $(-\infty, -a] \cup [a, \infty]$. Then a is equal to : **[Sep. 02, 2020 (I)]**

(a) $\frac{\sqrt{17}}{2}$ (b) $\frac{\sqrt{17}-1}{2}$ (c) $\frac{1+\sqrt{17}}{2}$ (d) $\frac{\sqrt{17}}{2}+1$

3. If $R = \{(x, y) : x, y \in \mathbf{Z}, x^2 + 3y^2 \le 8\}$ is a relation on the set of integers $\mathbf{Z}$, then the domain of R^{-1} is : **[Sep. 02, 2020 (I)]**

(a) $\{-2, -1, 1, 2\}$ (b) $\{0, 1\}$
(c) $\{-2, -1, 0, 1, 2\}$ (d) $\{-1, 0, 1\}$

4. Let $f: R \to R$ be defined by $f(x) = \frac{x}{1+x^2}$, $x \in R$. Then the range of f is : **[Jan. 11, 2019 (I)]**

(a) $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (b) $R - [-1, 1]$
(c) $R - \left[-\frac{1}{2}, \frac{1}{2}\right]$ (d) $(-1, 1) - \{0\}$

5. The domain of the definition of the function $f(x) = \frac{1}{4-x^2} + \log_{10}(x^3 - x)$ is: **[April. 09, 2019 (II)]**

(a) $(-1, 0) \cup (1, 2) \cup (3, \infty)$
(b) $(-2, -1) \cup (-1, 0) \cup (2, \infty)$
(c) $(-1, 0) \cup (1, 2) \cup (2, \infty)$
(d) $(1, 2) \cup (2, \infty)$

6. The range of the function

$f(x) = \frac{x}{1+|x|}$, $x \in R$, is **[Online May 7, 2012]**

(a) R (b) $(-1, 1)$ (c) $R - \{0\}$ (d) $[-1, 1]$

7. The domain of the function $f(x) = \frac{1}{\sqrt{|x| - x}}$ is **[2011]**

(a) $(0, \infty)$ (b) $(-\infty, 0)$
(c) $(-\infty, \infty) - \{0\}$ (d) $(-\infty, \infty)$

8. Domain of definition of the function

$f(x) = \frac{3}{4-x^2} + \log_{10}(x^3 - x)$, is **[2003]**

(a) $(-1,0) \cup (1,2) \cup (2,\infty)$ (b) $(a, 2)$
(c) $(-1,0) \cup (a,2)$ (d) $(1,2) \cup (2,\infty)$

TOPIC 2 Even and Odd Functions, Explicit and Implicit Functions, Greatest Integer Function, Periodic Functions, Value of a Function, Equal Functions, Algebraic Operations on Functions

9. Let $[t]$ denote the greatest integer $\le t$. Then the equation in x, $[x]^2 + 2[x+2] - 7 = 0$ has : **[Sep. 04, 2020 (I)]**

(a) exactly two solutions
(b) exactly four integral solutions
(c) no integral solution
(d) infinitely many solutions

10. Let $f(x)$ be a quadratic polynomial such that $f(-1)+f(2)=0$. If one of the roots of $f(x)=0$ is 3, then its other root lies in : **[Sep. 02, 2020 (II)]**

(a) $(-1, 0)$ (b) $(1, 3)$ (c) $(-3, -1)$ (d) $(0, 1)$

11. Let $f(1, 3) \to R$ be a function defined by $f(x)=\frac{x[x]}{1+x^2}$, where $[x]$ denotes the greatest integer $\le x$. Then the range of f is: **[Jan. 8, 2020 (II)]**

(a) $\left(\frac{2}{5}, \frac{3}{5}\right] \cup \left(\frac{3}{4}, \frac{4}{5}\right)$ (b) $\left(\frac{2}{5}, \frac{1}{2}\right) \cup \left(\frac{3}{5}, \frac{4}{5}\right]$

(c) $\left(\frac{2}{5}, \frac{4}{5}\right]$ (d) $\left(\frac{3}{5}, \frac{4}{5}\right)$

12. If $f(x)=\log_e\left(\frac{1-x}{1+x}\right)$, $|x|<1$, then $f\left(\frac{2x}{1+x^2}\right)$ is equal to : **[April 8, 2019 (I)]**

(a) $2f(x)$ (b) $2f(x^2)$ (c) $(f(x))^2$ (d) $-2f(x)$

13. Let $f(x)=a^x$ $(a>0)$ be written as $f(x)=f_1(x)+f_2(x)$, where $f_1(x)$ is an even function and $f_2(x)$ is an odd function. Then $f_1(x+y)+f_1(x-y)$ equals : **[April. 08, 2019 (II)]**

(a) $2f_1(x)f_1(y)$ (b) $2f_1(x+y)f_1(x-y)$

(c) $2f_1(x)f_2(y)$ (d) $2f_1(x+y)f_2(x-y)$

14. Let $f(n)=\left[\frac{1}{3}+\frac{3n}{100}\right]n$, where [n] denotes the greatest integer less than or equal to n. Then $\sum_{n=1}^{56} f(n)$ is equal to: **[Online April 19, 2014]**

(a) 56 (b) 689 (c) 1287 (d) 1399

15. Let f be an odd function defined on the set of real numbers such that for $x \ge 0$, $f(x) = 3\sin x + 4\cos x$.

Then f(x) at $x=-\frac{11\pi}{6}$ is equal to: **[Online April 11, 2014]**

(a) $\frac{3}{2}+2\sqrt{3}$ (b) $-\frac{3}{2}+2\sqrt{3}$

(c) $\frac{3}{2}-2\sqrt{3}$ (d) $-\frac{3}{2}-2\sqrt{3}$

16. A real valued function $f(x)$ satisfies the functional equation $f(x-y)=f(x)f(y)-f(a-x)f(a+y)$ where a is a given constant and $f(0)=0$, $f(2a-x)$ is equal to **[2005]**

(a) $-f(x)$ (b) $f(x)$

(c) $f(a)+f(a-x)$ (d) $f(-x)$

17. The graph of the function $y=f(x)$ is symmetrical about the line $x=2$, then **[2004]**

(a) $f(x)=-f(-x)$ (b) $f(2+x)=f(2-x)$

(c) $f(x)=f(-x)$ (d) $f(x+2)=f(x-2)$

18. If $f: R \to R$ satisfies $f(x+y)=f(x)+f(y)$, for all $x, y \in R$ and $f(1)=7$, then $\sum_{r=1}^{n} f(r)$ is **[2003]**

(a) $\frac{7n(n+1)}{2}$ (b) $\frac{7n}{2}$

(c) $\frac{7(n+1)}{2}$ (d) $7n+(n+1)$

Hints & Solutions

1. **(a)** For R_1 let $a = 1+\sqrt{2},\ b = 1-\sqrt{2},\ c = 8^{1/4}$

$aR_1b \Rightarrow a^2 + b^2 = (1+\sqrt{2})^2 + (1-\sqrt{2})^2 = 6 \in Q$

$bR_1c \Rightarrow b^2 + c^2 = (1-\sqrt{2})^2 + (8^{1/4})^2 = 3 \in Q$

$aR_1c \Rightarrow a^2 + c^2 = (1+\sqrt{2})^2 + (8^{1/4})^2 = 3 + 4\sqrt{2} \notin Q$

$\therefore R_1$ is not transitive.

For R_2 let $a = 1+\sqrt{2},\ b = \sqrt{2},\ c = 1-\sqrt{2}$

$aR_2b \Rightarrow a^2 + b^2 = (1+\sqrt{2})^2 + (\sqrt{2})^2 = 5 + 2\sqrt{2} \notin Q$

$bR_2c \Rightarrow b^2 + c^2 = (\sqrt{2})^2 + (1-\sqrt{2})^2 = 5 - 2\sqrt{2} \notin Q$

$aR_2c \Rightarrow a^2 + c^2 = (1+\sqrt{2})^2 + (1-\sqrt{2})^2 = 6 \in Q$

$\therefore R_2$ is not transitive.

2. **(c)** $\because f(x) = \sin^{-1}\left(\dfrac{|x|+5}{x^2+1}\right)$

$\therefore -1 \le \dfrac{|x|+5}{x^2+1} \le 1$

$\Rightarrow |x| + 5 \le x^2 + 1 \qquad [\because x^2 + 1 \ne 0]$

$\Rightarrow x^2 - |x| - 4 \ge 0$

$\Rightarrow \left(|x| - \dfrac{1-\sqrt{17}}{2}\right)\left(|x| - \dfrac{1+\sqrt{17}}{2}\right) \ge 0$

$\Rightarrow x \in \left(-\infty, -\dfrac{1+\sqrt{17}}{2}\right) \cup \left[\dfrac{1+\sqrt{17}}{2}, \infty\right)$

$\therefore a = \dfrac{1+\sqrt{17}}{2}$

3. **(d)** Since, $R = \{(x, y) : x, y \in \mathbf{Z},\ x^2 + 3y^2 \le 8\}$

$\therefore R = \{(1, 1), (2, 1), (1, -1), (0, 1), (1, 0)\}$

$\Rightarrow D_{R^{-1}} = \{-1, 0, 1\}$

4. **(a)** $f(x) = \dfrac{x}{1+x^2}, x \in R$

Let, $y = \dfrac{x}{1+x^2}$

$\Rightarrow\ yx^2 - x + y = 0 \Rightarrow x = \dfrac{1 \pm \sqrt{1-4y^2}}{2}$

$\Rightarrow\ 1 - 4y^2 \ge 0$

$\Rightarrow\ 1 \ge 4y^2$

$\Rightarrow\ |y| \le \dfrac{1}{2}$

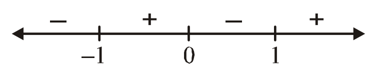
$\Rightarrow\ -\dfrac{1}{2} \le y \le \dfrac{1}{2}$

$\Rightarrow$ The range of f is $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$.

5. **(c)** To determine domain, denominator $\ne 0$ and $x^3 - x > 0$

i.e., $4 - x^2 \ne 0\ x \ne \pm 2$ ...(1)

and $x(x-1)(x+1) > 0$



$x \in (-1, 0) \cup (1, \infty)$ ...(2)

Hence domain is intersection of (1) & (2).

i.e., $x \in (-1, 0) \cup (1, 2) \cup (2, \infty)$

6. **(b)** $f(x) = \dfrac{x}{1+|x|}, x \in R$

If $x > 0, |x| = x \Rightarrow f(x) = \dfrac{x}{1+x}$

which is not defined for $x = -1$

If $x < 0, |x| = -x \Rightarrow f(x) = \dfrac{x}{1-x}$

which is not defined for $x = 1$

Thus $f(x)$ defined for all values of R except 1 and – 1

Hence, range = (– 1, 1).

7. **(b)** $f(x) = \dfrac{1}{\sqrt{|x| - x}}$, $f(x)$ is define if $|x| - x > 0$

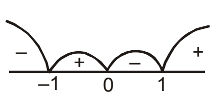
$\Rightarrow |x| > x, \Rightarrow x < 0$

Hence domain of $f(x)$ is $(-\infty, 0)$

8. **(a)** $f(x) = \dfrac{3}{4-x^2} + \log_{10}(x^3 - x)$

$4 - x^2 \ne 0;\ x^3 - x > 0;$

$x \ne \pm\sqrt{4}$ and $-1 < x < 0$ or $1 < x < \infty$



$\therefore D = (-1, 0) \cup (1, \infty) - \{\sqrt{4}\}$

$D = (-1, 0) \cup (1, 2) \cup (2, \infty)$.

9. **(d)** The given equation

$[x]^2+2[x]+4-7=0$

$\Rightarrow [x]^2+2[x]-3=0$

$\Rightarrow [x]^2+3[x]-[x]-3=0$

$\Rightarrow ([x]+3)([x]-1)=0 \Rightarrow [x]=1$ or -3

$\Rightarrow x \in [-3,-2) \cup [1, 2)$

$\therefore$ The equation has infinitely many solutions.

10. **(a)** Let $f(x)=ax^2+bx+c$

Given : $f(-1)+f(2)=0$

$a-b+c+4a+2b+c=0$

$\Rightarrow 5a+b+2c=0$ ...(i)

and $f(3)=0 \Rightarrow 9a+3b+c=0$ ...(ii)

From equations (i) and (ii),

$$\frac{a}{1-6}=\frac{b}{18-5}=\frac{c}{15-9} \Rightarrow \frac{a}{-5}=\frac{b}{13}=\frac{c}{6}$$

Product of roots, $\alpha\beta=\frac{c}{a}=\frac{-6}{5}$ and $\alpha=3$

$$\Rightarrow \beta=\frac{-2}{5}\in(-1,0)$$

11. **(b)** $f(x)\begin{cases}\dfrac{x}{x^2+1}; & x\in(1,2)\\ \dfrac{2x}{x^2+1}; & x\in[2,3)\end{cases}$

$$f'(x)\begin{cases}\dfrac{1-x^2}{1+x^2}; & x\in(1,2)\\ \dfrac{1-2x^2}{1+x^2}; & x\in[2,3)\end{cases}$$

$\therefore$ $f(x)$ is a decreasing function

$$\therefore \quad y\in\left(\frac{2}{5},\frac{1}{2}\right)\cup\left(\frac{6}{10},\frac{4}{5}\right]$$

$$\Rightarrow \quad y\in\left(\frac{2}{5},\frac{1}{2}\right)\cup\left(\frac{3}{5},\frac{4}{5}\right]$$

12. **(a)** $f(x)=\log\left(\frac{1-x}{1+x}\right), |x|<|$

$$f\left(\frac{2x}{1+x^2}\right)=\log\left(\frac{1-\frac{2x}{1+x^2}}{1+\frac{2x}{1+2x^2}}\right)$$

$$=\log\left(\frac{1+x^2-2x}{1+x^2+2x}\right) \quad =\log\left(\frac{1-x}{1+x}\right)^2$$

$$=2\log\left(\frac{1-x}{1+x}\right) \quad =2f(x)$$

13. **(a)** Given function can be written as

$$f(x)=a^x=\left(\frac{a^x+a^{-x}}{2}\right)+\left(\frac{a^x-a^{-x}}{2}\right)$$

where $f_1(x)=\dfrac{a^x+a^{-x}}{2}$ is even function

$f_2(x)=\dfrac{a^x-a^{-x}}{2}$ is odd function

$\Rightarrow f_1(x+y)+f_1(x-y)$

$$=\left(\frac{a^{x+y}+a^{-x-y}}{2}\right)+\left(\frac{a^{x-y}+a^{-x+y}}{2}\right)$$

$$=\frac{1}{2}\left[a^x(a^y+a^{-y})+a^{-x}(a^y+a^{-y})\right]$$

$$=\frac{(a^x+a^{-x})(a^y+a^{-y})}{2}=2f_1(x)\,.\,f_1(y)$$

14. **(d)** Let $f(n)=\left[\frac{1}{3}+\frac{3n}{100}\right]n$

where $[n]$ is greatest integer function,

$$=\left[0.33+\frac{3n}{100}\right]n$$

For n = 1, 2, ..., 22, we get $f(n)=0$

and for n = 23, 24, ..., 55, we get $f(n)=1\times n$

For n = 56, $f(n)=2\times n$

So, $\sum_{n=1}^{56} f(n)=1\,(23)+1\,(24)+\ldots+1\,(55)+2(56)$

$=(23+24+\ldots+55)+112$

$=\frac{33}{2}[46+32]+112$

$=\frac{33}{2}(78)+112=1399.$

15. **(c)** Given f be an odd function

$f(x)=3\sin x+4\cos x$

Now,

$$f\left(\frac{-11\pi}{6}\right)=3\sin\left(\frac{-11\pi}{6}\right)+4\cos\left(\frac{-11\pi}{6}\right)$$

$$f\left(\frac{-11\pi}{6}\right) = 3\sin\left(-2\pi+\frac{\pi}{6}\right)+4\cos\left(-2\pi+\frac{\pi}{6}\right)$$

$$f\left(\frac{-11\pi}{6}\right) = 3\sin\left\{-\left(2\pi-\frac{\pi}{6}\right)\right\}+4\cos\left\{-\left(2\pi-\frac{\pi}{6}\right)\right\}$$

$$\left\{\text{For odd functions } \begin{array}{l}\sin(-\theta) = -\sin\theta \\ \text{and} \cos(-\theta) = -\cos\theta\end{array}\right\}$$

$$\therefore\ f\left(\frac{-11\pi}{6}\right) = -3\sin\left(2\pi-\frac{\pi}{6}\right)-4\cos\left(2\pi-\frac{\pi}{6}\right)$$

$$\Rightarrow f\left(\frac{-11\pi}{6}\right) = +3\sin\left(\frac{\pi}{6}\right)-4\cos\frac{\pi}{6}$$

$$\Rightarrow f\left(\frac{-11\pi}{6}\right) = 3\times\frac{1}{2}-4\times\frac{\sqrt{3}}{2}$$

$$\text{or } f\left(\frac{-11\pi}{6}\right) = \frac{3}{2}-2\sqrt{3}$$

16. **(a)** Given that f(0) = 0 and put

$x = 0, y = 0,$

$f(0) = f^2(0) - f^2(a)$

$\Rightarrow f^2(a) = 0 \Rightarrow f(a) = 0$

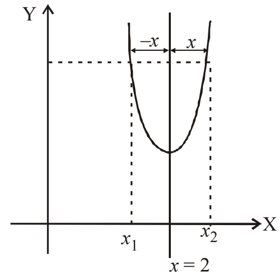
$f(2a-x) = f(a-(x-a))$

$= f(a)f(x-a) - f(0)f(x)$

$= f(a)f(x-a) - f(x) = -f(x)$

$\Rightarrow f(2a-x) = -f(x)$

17. **(b)** Given that a graph symmetrical. with respect to line $x = 2$ as shown in the figure.



From the figure

$f(x_1) = f(x_2)$, where $x_1 = 2 - x$ and $x_2 = 2 + x$

$\therefore f(2-x) = f(2+x)$

18. **(a)** $f(x+y) = f(x) + f(y)$.

$\because f(1) = 7$

$f(2) = f(1+1) = f(1) + f(1) = 14$

$f(3) = f(1+2) = f(1) + f(2) = 21$

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$$\therefore \sum_{r=1}^{n} f(r) = 7(1+2+3.....+n)$$

$$= \frac{7n(n+1)}{2}$$